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On product expansions of certain modular functions

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1. Introduction

As is well-known, of course, a meromorphic modular form on a subgroup of the full modular group $\Gamma_1 := SL_2(\mathbf{Z})$ of finite index has a Fourier expansion and the Fourier coefficients often are interesting arithmetical functions. One need only think of the number of respresentations of integers by quadratic forms (coefficients of theta series), the sums of powers of divisors of integers (coefficients of Eisenstein series) or the Ramanujan tau-function (coefficients of the discriminant function).

As may be less well-known, modular forms also have product expansions and the exponents in these expansions also seem to often carry important information on the function. For example, there is famous work by Borcherds [1] which relates the exponents of modular forms with so-called "Heegner divisors" to Fourier coefficients of modular forms of half-integral weight, and there is also work by Bruinier, Ono and the author [2] in which these exponents quite generally are related to special values of the usual modular invariant j at the points of the divisor of the function.

In this short survey articel, I would like to report on recent work and point out that these exponents can also be used to give a characterization of those modular forms which do not have zeros or poles on the upper half-plane. For details the reader is referred to [5,6].

2. Product expansions of modular forms

Recall that if $\Gamma \subset \Gamma_1$ is a subgroup of finite index and $f : \mathcal{H} \to \mathbf{C}$ is a modular form of integral weight k on Γ (where $\mathcal{H}$ denotes the complex upper half-plane), then f has a Fourier expansion

$$f(z) = \sum_{n \geq h} a(n) q_M^n \qquad (0 < |q_M| < \epsilon)$$

for some $h \in \mathbf{Z}$ and $\epsilon > 0$. Here we have written $q_M = e^{2\pi i z/M}$ $(z \in \mathcal{H})$ and M is the least positive integer with $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Lemma [3]. *Let $f(z) = \sum_{n \geq h} a(n) q_M^n$ be a function meromorphic at $q_M = 0$, with $a(h) \neq 0$. Then there are uniquely determined complex numbers $c(n)$ $(n \in \mathbf{N})$ such that*

$$f(z) = a(h) q_M^h \prod_{n \geq 1} (1 - q_M^n)^{c(n)}$$

where the infinite product converges for $|q_M| < \delta$ for some $\delta > 0$ and complex powers are defined as usual by the principal branch of the complex logarithm. Moreover, if

$$\theta = q_M \frac{d}{dq_M} = \frac{M}{2\pi i}\frac{d}{dz},$$

then

$$\frac{\theta f}{f} = h - \sum_{n\geq 1}\Big(\sum_{d|n} dc(d)\Big)q_M^n \qquad (|q_M| < \delta).$$

Examples: i) $\Delta = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$ (discriminant function of weight 12 on Γ_1; $q = q_1$).

ii) $E_4 = 1 + 240\sum_{n\geq 1}(\sum_{d|n,d>0} d^3)q^n$ (Eisenstein series of weight 4 on Γ_1). One has

$$E_4 = (1-q)^{-240}(1-q^2)^{26760}\ldots = \prod_{n\geq 1}(1-q^n)^{c(n)}$$

where $c(n) = b(n^2)$ $(n \in \mathbf{N})$ and

$$\sum_{n\geq -3, n\equiv 0,1 \pmod 4} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 \pm \ldots$$

is a certain meromorphic modular form of weight $\frac{1}{2}$ and level 4.

This follows from Borcherds' theory [1] since the unique zero modulo Γ_1 of E_4 is $z = e^{2\pi i/3}$ which is a "Heegner point". (Recall that in general $z \in \mathcal{H}$ is called a "Heegner point" if it satisfies a quadratic equation $az^2 + bz + c = 0$ with $a, b, c \in \mathbf{Z}$, $gcd(a, b, c) = 1$ and $b^2 - 4ac < 0$.)

The general experience is that if the Fourier coefficients are simple, then the exponents are mysterious, and converseley.

3. A characterization of forms with divisors supported at the cusps

Let $f \neq 0$ be a meromorphic modular form of weight $k \in \mathbf{Z}$ on $\Gamma \subset \Gamma_1$.

Theorem 1 [5]. *Suppose that f has no zeros or poles on $\mathcal{H}$. Then this assertion is equivalent to the following assertions, respectively:*

i) if Γ is of finite index in Γ_1, then $c(n) <<_f \log\log n \cdot \log n$ $(n > 2)$ where the constant implied in $<<_f$ only depends on f;

ii) if Γ is a congruence subgroup of Γ_1, then $c(n) <<_f (\log\log n)^2$ $(n > 2)$ where the constant implied in $<<_f$ only depends on f.

In case f is on $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid N|c\}$ for some $N \in \mathbf{N}$, one can do much better (at least if N is squarefree) and prove

Theorem 2 [5]. *Suppose that $\Gamma = \Gamma_0(N)$ and N is squarefree. Then f has no zeros or poles on $\mathcal{H}$ if and only if $c(n)\,(n \in \mathbf{N})$ depends only on the greatest common divisor $gcd(n, N)$.*

Theorem 2 fails if one merely assumes that f is a generalized modular form in the sense of Knopp-Mason [4], i.e. in the usual definition of a modular form one essentially drops the assumption that the multiplier system is of absolute value 1.

Theorem 3 [6]. *Let k and N be integers with $N \geq 11$ and $12|k$. Then there is a generalized modular form f of weight k on $\Gamma_0(N)$ such that*

i) f has no zeros on $\mathcal{H}$;
ii) the exponents $c(n)\,(n \in \mathbf{N})$ take infinitely many different values.

For the proofs of the above results we refer to [5,6].

References

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